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A relevant two qubit Bell inequality inequivalent to the CHSH inequality

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Abstract

We computationally investigate the complete polytope of Bell inequalities for two particles with small numbers of possible measurements and outcomes. Our approach is limited by Pitowsky's connection of this problem to the computationally hard NP problem. Despite this, we find that there are very few relevant inequivalent inequalities for small numbers. For example, in the case with three possible 2-outcome measurements on each particle, there is just one new inequality. We describe mixed 2-qubit states which violate this inequality but not the CHSH. The new inequality also illustrates a sharing of bi-partite non-locality between three qubits: something not seen using the CHSH inequality. It also inspires us to discover a class of Bell inequalities with m possible n -outcome measurements on each particle.

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1. Introduction

How are actions and events in different places connected to one another? Normally we imagine that the correlations were arranged in the past. Both my socks are black since I put on a pair this morning. However our quantum mechanical theory of the world is more complicated [1]. Correlations are created in at least one more way. Some possibilities are that (a) correlations are arranged through faster than light influences in the present, (b) correlations are arranged in a many world scenario, (c) correlations just occur: they constitute a primary concept, preventing us from consistently thinking about local subsystems. Since these possibilities are disliked by many physicists, we have studied the set of correlations which can be generated by the past, and when quantum mechanics goes beyond this. Our goal is first to find a simple set of conditions—generalized Bell inequalities—which describe the boundaries of the set of past-generated correlations. This set is often called the set of common cause correlations, local hidden variable (lhv) correlations, local variable correlations, or local realistic correlations. Our second goal is to see when quantum mechanics goes outside this set.

One reason for studying this boundary so closely is the fundamental question: which correlations can be generated in this way? A second reason is that a violation of Bell inequalities gives a signature for useful entanglement. For instance, violation of a certain Bell inequality by an N qubit state implies that the state is distillable [2]: perfect bipartite entanglement can be extracted from it. Also, Bell inequalities can be used as a simple test for the security of quantum cryptography [3]. The rough idea is that past-generated correlations could have been created (and thus known) by an eavesdropper, and so are not useful for cryptography, whereas other kinds of correlations cannot be created by the eavesdropper and so are useful. The connection holds very closely in some of the main cases of interest [4].

Bell inequalities are also related to classical communication complexity [5]: how much communication do two parties need in order to perform some joint task? A final reason is that no experiment has definitively demonstrated correlations outside the past-generated set [6]. We hope to find one day an inequality which will allow us to do this without waiting for improved technology.

A typical experiment to test for correlations begins by creating a particular quantum mechanical state of two particles, and sending one to site A , and the other to site B . We then perform one of a certain number, m_A and m_B say, of possible measurements, i_A and i_B , at each of the two sites, A and B . Each measurement has a certain number, n_A and n_B , of possible outcomes, j_A and j_B . We then repeat the experiment many times to get accurate probabilities for each set of joint outcomes, $P(j_A, j_B | i_A, i_B)$. The $d \equiv m_A m_B n_A n_B$ probabilities can be thought of as a point in a d -dimensional space. We are interested in the set of all points which can be described using past-generated correlations. This set is convex, and the boundary is defined by hyperplanes. It is straightforward to list the $n_A^{m_A} n_B^{m_B}$ vertices. We would like to know the faces, otherwise known as the Bell inequalities. We want to know how many different types of faces there are, and moreover, which ones are relevant for quantum mechanics.

Characterising the set of past-generated correlations is difficult. Here difficult is meant in a technical sense. Suppose we are given a point and asked if it is in the set. Finding the answer is a finite computation, but will take more time with more possible measurements, $m_A m_B$. In fact, Pitowsky has shown this problem to be NP-complete [7]. Furthermore, suppose we are given an inequality, and wish to know whether or not it is a face. This problem is of similar difficulty: co-NP complete [8].

Since the general problem is so hard, we calculated the Bell inequalities for various small numbers of measurements and outcomes. Surprisingly, we have found that for small numbers of measurements and outcomes there are very few inequivalent Bell inequalities. For the case ($m_A = 2, m_B = 2, n_A = 2, n_B = 2$) (hereafter called 2222), Fine [9] has shown that up to certain equivalences which we shall describe below, the only Bell inequality is the CHSH [10]. For the case 3322 we have found that there is only a single new inequality. Considering the complexity of the problem, this is very surprising and simple. Furthermore, the inequality is relevant, since there are states which violate it but do not violate the CHSH inequality. We believe this is the first time that performing more than two measurements has been shown to be useful for detecting non-local correlations.

The confidence gained from such results (we shall describe more) has also helped us to generalize this inequality to the case $mmnn$. Thus the computational approach, whilst limited, has proved rather useful. However we do not know if these inequalities are relevant: are there states which do not violate the previous inequalities which violate these new ones? Computationally, it seems the answer is 'no', though we have no systematic method for checking this. But if these inequalities are not relevant, this would further reduce the number of inequivalent, relevant Bell inequalities.

2. The space of classical correlations

Before describing our results, we must describe more precisely the set of correlations, and what we mean by equivalent and relevant inequalities. As previously stated, the correlations live in a d -dimensional space with components representing $P(j_A, j_B|i_A, i_B)$. Some of these components are redundant, however. For example, for any fixed measurement, there has to be at least one outcome, i.e. $\sum_{j_A, j_B} P(j_A, j_B|i_A, i_B) = 1$. Also the correlations we are interested in, those of quantum (and classical) mechanics, do not allow faster than light signalling. In other words, the distribution of outcomes on one particle does not depend upon the choice of measurement made on the other: $\sum_{j_A} P(j_A, j_B|i_A, i_B) = \sum_{j_A} P(j_A, j_B|i'_A, i_B)$. Therefore we work in the subspace of dimension

$$d_2 \equiv m_A m_B (n_A - 1)(n_B - 1) + m_A (n_A - 1) + m_B (n_B - 1) \quad (1)$$

which satisfies all of these constraints. Our subspace can be labelled using the components $P(j_A, j_B|i_A, i_B)$ for $j_A = 0..(n_A - 2)$, $j_B = 0..(n_B - 2)$, $i_A = 1..m_A$, $i_B = 1..m_B$; the components $P(j_A|i_A)$ for $j_A = 0..(n_A - 2)$, $i_A = 1..m_A$; and the components $P(j_B|i_B)$ for $j_B = 0..(n_B - 2)$, $i_B = 1..m_B$.

We are interested in the faces of the convex set in this reduced space. These are ‘tight’ inequalities. There are of course other inequalities which are satisfied by all the points inside the past-generated set, but such inequalities are less useful for detecting non-local correlations. An inequality will describe a $(d_2 - 1)$ -dimensional hyperplane, and so has d_2 components.

Some of these faces are equivalent. For example, when defining the experiment, ‘we have to decide which particle is A and which is B . We also have to assign the labels j_A to the outcomes of measurements on particle A , the labels j_B to the outcomes of measurements on particle B , and the labels i_A and i_B to the different measurements we could perform on particles A and B ’. Since these choices are arbitrary, we shall consider two inequalities to be equivalent if they can be converted into one another simply by relabelling these local choices.

Having found the inequivalent faces, we would like to know if they are violated by quantum mechanics. As our motivation for studying bell inequalities comes from quantum physics, we are interested only in the faces which are violated.

Given two inequalities which are violated, we define the first to be *non-redundant* if quantum mechanics gives a point $P(j_A, j_B|i_A, i_B)$ which violates the first inequality, but which does not violate the second inequality (or any inequality equivalent to the second). Given several inequalities, we could look for the minimal set of inequalities such that none are redundant.

We are more interested in a classification which comes from quantum mechanical states. Given two inequalities we define the first to be *relevant* if there exists a quantum state which violates it for some choice of measurements, but does not violate the second inequality for any choice of measurements. Similarly for the second inequality. Given a set of inequalities, we want to find the minimal set of relevant ones. Note that this will lead to a minimal set no larger than the set of non-redundant inequalities: in fact it may be much smaller.

One is often interested in quantum systems of a certain size, like 2-qubit systems. How many relevant inequalities are there for 2-qubits? Before the present work, only one—the CHSH—was known to be relevant. An open question was whether more measurements (or more outcomes) could help. Here we show that inequalities with three measurements are relevant. We do not know whether four or more will help.

More outcomes may also be useful even on qubits, since we could perform a several outcome Positive Operator Valued Measurement (POVM). Whilst the usefulness remains an open question, we can at least put an upper bound of d^2 outcomes for any measurement on

a d -dimensional system. Thus for 2-qubits, it is not useful to have more than four outcomes for any one measurement. The reason for this is that a POVM in a d^2 -dimensional space (the dimension of the density matrix) can always be viewed as a classical probabilistic mixture of POVM with d^2 outcomes [11]. In other words, the many outcome POVM can be viewed in two stages. The first rolls an independent dice to decide which few-outcome measurement to make. The second performs the few outcome measurement, and gives the appropriate outcome. Adding local randomness which is under the control of the lhv model cannot add non-locality, and so any non-locality present in such a many outcome measurement must already be there in the few outcome measurements, and hence in a few outcome inequality.

The limit on the number of useful outcomes is also interesting since it suggests there are inequalities which are irrelevant for 2-qubits, but which are useful for higher dimensional systems. For example the five-dimensional Collins–Gisin–Linden–Massar–Popescu (CGLMP) inequality [12], which deals with the case 2255, is irrelevant for qubits (it has too many outcomes). On the other hand, there are five-dimensional states which violate this inequality which are not known to violate any lower dimensional inequalities. With our present knowledge, this inequality is indeed useful for such systems.

In order to find all the inequivalent inequalities, we have several software tools. The main one is a linear programming tool which takes a list of vertices as input and, after some time, outputs all the faces¹. We have written a small matlab program which, given m_A, m_B, n_A, n_B , produces a list of the vertices. The vertices are given by distributions which factor into two local probability distributions, i.e.

$$P(j_A, j_B | i_A, i_B) = P(j_A | i_A) P(j_B | i_B) \quad (2)$$

and for which all the local probabilities are either 0 or 1, e.g. for 2222,

$$P(j_A = 0 | i_A = 1) = 0 \quad (3)$$

$$P(j_A = 0 | i_A = 2) = 1 \quad (4)$$

$$P(j_B = 0 | i_B = 1) = 1 \quad (5)$$

$$P(j_B = 0 | i_B = 2) = 0. \quad (6)$$

After we have the list of faces, we put this into a second matlab program which removes the equivalences, leaving us with the inequivalent inequalities. These software are all deterministic, and so give the exact solution. The bottleneck is the freely available linear programming tool, which is optimized for certain kinds of convex sets, but not for the equivalences which we have here. We have a final piece of software² which, given an inequality and either the size of the quantum system (e.g. 2-qubits) or a specific quantum state, probabilistically finds the maximum value of the inequality.

3. The Bell inequalities

For the case 2222 the software reproduces Fine's result that there are only two types of inequality. One is the trivial one that probabilities are positive, i.e. $P(j_A, j_B | i_A, i_B) \geq 0$. This occurs $m_A m_B n_A n_B = 16$ times (to cover all the joint probabilities). There is no need for

¹ This is part of a package called Polymake, available from www.math.tu-berlin.de/polymake. It includes several different algorithms for solving the linear programming problem. The fastest for our problems is Porta, available at www.zib.de/Optimization/Software/Porta. As an alternative we used cdd, which is included in Polymake or from www.cs.mcgill.ca/~fukuda/soft/cdd_home/cdd.html.

² Thanks to Bernard Gisin for this software, e-mail address: Bernard.Gisin@Infomaniak.ch.

inequalities stating that probabilities should be not greater than 1, since this follows from all the probabilities being positive. The other type of inequality is the Clauser–Horne–Shimony–Holt (CHSH), which occurs eight times. We write it here in a form closer to that of the CH inequality [13]

$$I_{\text{CHSH}} = P(A_1 B_1) + P(A_2 B_1) + P(A_1 B_2) - P(A_2 B_2) - (P(A_1) + P(B_1)) \tag{7}$$

where $P(AB)$ is the probability that when A and B are measured we get the outcome 0 for both measurements. $I_{\text{CHSH}} \leq 0$ for lhv correlations. Quantum mechanics can attain values up to $\frac{1}{\sqrt{2}} - \frac{1}{2}$.

It will be useful for later on to write this inequality in the following way:

$$I_{\text{CHSH}} = \left(\begin{array}{c|cc} & -1 & 0 \\ \hline -1 & 1 & 1 \\ 0 & 1 & -1 \end{array} \right) \tag{8}$$

where the table gives the coefficients we are to put in front of the probabilities:

$$\left(\begin{array}{c|cc} & P(A_1) & P(A_2) \\ \hline P(B_1) & P(A_1 B_1) & P(A_2 B_1) \\ P(B_2) & P(A_1 B_2) & P(A_2 B_2) \end{array} \right). \tag{9}$$

Next we computed the case 2322. This is a choice between two 2-outcome measurements on one particle, and between three 2-outcome measurements on the other particle. Here we found no new inequalities. We have only that the probabilities must be positive, and CHSH inequalities where the results of one of the three measurements are ignored, e.g.

$$I'_{\text{CHSH}} = P(A_1 B_2) + P(A_2 B_2) + P(A_1 B_3) - P(A_2 B_3) - (P(A_1) + P(B_2)). \tag{10}$$

There are $m_A m_B n_A n_B = 24$ ‘positive probability’ faces, and $8\binom{3}{2} = 24$ CHSH faces—we have to choose two of the three possible measurements for B, and once these are chosen we have the eight versions of the CHSH inequality which appear in the 2222 case. This gives a total of 48 faces.

We have analytically extended this result to the case 2m22. We find that there are no new inequalities for this case. Our proof is essentially to note that the proof of Fine [9] for the 2222 case extends naturally to the 2m22 case. Fine’s proof works by starting with the measured probabilities $P(j_A, j_B | i_A, i_B)$, which are assumed to satisfy the CHSH inequalities. He then constructs a lhv model which reproduces the measured probabilities. We shall describe the construction for the 3222 case: the general m222 case follows very naturally.

First we define β to be the minimum of eight quantities: $P(B_1)$, $P(A_1 B_1) + P(B_2) - P(A_1 B_2)$, and the other six quantities which come from exchanging A_1 for A_2 or A_3 and B_1 for B_2 in the previous expressions. We set $P(B_1, B_2) \equiv \beta$. $P(B_1)$ and $P(B_2)$ are experimentally measurable so we can complete the distribution for B_1 and B_2 by $P(B_1, \bar{B}_2) \equiv P(B_1) - \beta$, $P(\bar{B}_1, B_2) \equiv P(B_2) - \beta$ and $P(\bar{B}_1, \bar{B}_2) = 1 - P(B_1) - P(B_2) + \beta$, where $P(\bar{B}) \equiv P(B = 1)$. One can check that all these probabilities are positive, using the fact that all the measured probabilities $P(j_A, j_B | i_A, i_B)$ are positive.

We extend this to a lhv model for A_1, B_1 and B_2 . We define α to be the minimum of $P(A_1, B_1)$, $P(A_1, B_2)$, β and $\beta - (P(A_1) + P(B_1) + P(B_2) - P(A_1, B_1) - P(A_1, B_2) - 1)$. We set $P(A_1, B_1, B_2) \equiv \alpha$. We can check this is well defined using the CHSH inequalities. We complete the distribution for (A_1, B_1, B_2) using the quantities we already have. i.e.

$$P(A_1, B_1, \bar{B}_2) \equiv P(A_1, B_1) - \alpha \tag{11}$$

$$P(A_1, \bar{B}_1, B_2) \equiv P(A_1, B_2) - \alpha \tag{12}$$

$$P(A_1, \bar{B}_1, \bar{B}_2) \equiv P(A_1) - P(A_1, B_1) - P(A_1, B_2) + \alpha \tag{13}$$

$$P(\bar{A}_1, B_1, B_2) \equiv \beta - \alpha \tag{14}$$

$$P(\bar{A}_1, B_1, \bar{B}_2) \equiv P(B_1) - P(A_1, B_1) - (\beta - \alpha) \tag{15}$$

$$P(\bar{A}_1, \bar{B}_1, B_2) \equiv P(B_2) - P(A_1, B_2) - (\beta - \alpha) \tag{16}$$

$$P(\bar{A}_1, \bar{B}_1, \bar{B}_2) \equiv P(A_1, B_1) + P(A_1, B_2) + (\beta - \alpha) + 1 - P(A_1) - P(B_1) - P(B_2). \tag{17}$$

That these are all positive follows from the CHSH inequalities.

In a similar way, we make lhv models for the triple (A_2, B_1, B_2) and the triple (A_3, B_1, B_2) .

We finally extend this to a distribution for $(A_1, A_2, A_3, B_1, B_2)$ by

$$P(A_1, A_2, A_3, B_1, B_2) \equiv P(A_1|B_1, B_2)P(A_2|B_1, B_2) * P(A_3|B_1, B_2)P(B_1, B_2) \tag{18}$$

where $P(A_1|B_1, B_2) = P(A_1, B_1, B_2)/P(B_1, B_2)$. This gives us a well-defined lhv distribution which reproduces all the measured probabilities.

4. A relevant new inequality for qubits

For three possible measurements on each side, the case 3322, Garg and Mermin [14] have shown that the CHSH inequalities are not the only faces of the classical polytope. They found a point which satisfies all the CHSH inequalities, but does not admit a lhv model. A complete list of the faces have been computed by Pitowsky and Svozil [15]. They found 684 faces. Removing equivalent inequalities, we find a single new inequality. The 684 faces of the 3322 polytope are made up as $m_{AMBNANB} = 36$ ‘positive probability’ faces, $8\binom{3}{2}^2 = 72$ CHSH faces and 576 equivalent new faces. The new face is

$$I_{3322} = \left(\begin{array}{c|ccc} & -1 & 0 & 0 \\ \hline -2 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{array} \right). \tag{19}$$

This expression satisfies $I_{3322} \leq 0$ for past-generated correlations. For quantum mechanics a numerical optimization suggests that the maximum value is 0.25. This value can be attained by the maximally entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0, 1\rangle - |1, 0\rangle). \tag{20}$$

The measurements all lie in a plane, so we denote their position by a single angle, the angle which they make with the z -axis in the Bloch sphere. $A_1 = 0, A_2 = \frac{\pi}{3}, A_3 = \frac{2\pi}{3}, B_1 = \frac{4\pi}{3}, B_2 = \pi$ and $B_3 = \frac{2\pi}{3}$.

The most interesting feature of this inequality is that there exist states which violate it which do not violate the CHSH inequality. For example, consider the 2-qubit state

$$\sigma = 0.85P_{|\phi\rangle} + 0.15P_{|0,1\rangle} \tag{21}$$

where $P_{|\phi\rangle}$ is the projector onto the state $|\phi\rangle$, and

$$|\phi\rangle = \frac{1}{\sqrt{3}}(2|0, 0\rangle + |1, 1\rangle). \tag{22}$$

One can check (using the Horodecki criterion [16]) that this state does not violate the CHSH inequality. However it does violate the 3322 inequality, giving a value ~ 0.0129 . The measurements for this violation are Von-Neumann measurements in the directions $(\theta_{\text{azim}}, \theta_{\text{polar}})$, where θ_{azim} is the azimuthal angle with the z -axis, and θ_{polar} is the polar angle in the x - y plane, when we set the z -axis to be in the direction of $|0\rangle$, and the x -axis to be in the

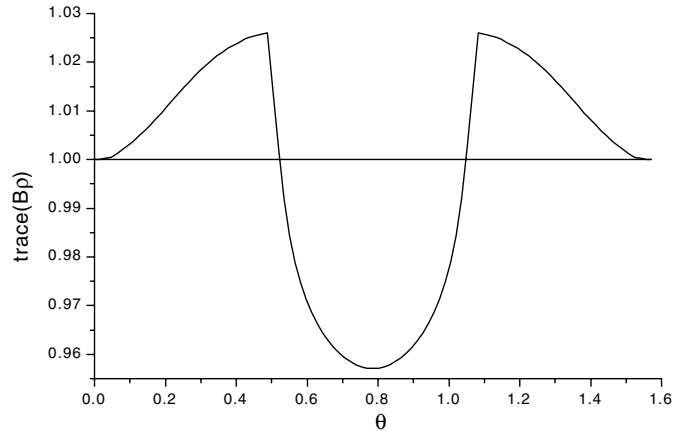


Figure 1. Maximum value of two Bell inequalities for a family of states. The straight horizontal line is that of \tilde{I}_{CHSH} , whilst the curve is for I_{3322} .

direction of $|0\rangle + |1\rangle$:

$$A_1 = (\eta, 0) \tag{23}$$

$$A_2 = (-\eta, 0) \tag{24}$$

$$A_3 = (-\pi/2, 0) \tag{25}$$

$$B_1 = (-\chi, 0) \tag{26}$$

$$B_2 = (\chi, 0) \tag{27}$$

$$B_3 = (\pi, 2) \tag{28}$$

where $\cos \eta = \sqrt{(7/8)}$ and $\cos \chi = \sqrt{(2/3)}$.

To compare the inequalities CHSH and I_{3322} , we numerically calculated the maximum violation, $\text{Tr}(B\rho)$, of I_{3322} for all possible Von-Neumann measurements for a family of states parametrized by θ ,

$$\rho_\theta = \lambda_{\text{CHSH}} P_{\cos \theta|0,0\rangle + \sin \theta|1,1\rangle} + (1 - \lambda_{\text{CHSH}}) P_{|0,1\rangle} \tag{29}$$

where λ_{CHSH} is chosen so that each state in the family ρ_θ gives the maximal value of the CHSH inequality which can be obtained by lhv theories. In order to give some meaning to the size of the violation, we have rescaled I_{CHSH} and I_{3322} so that the lhv maximum is 1, and the maximally mixed state $\rho = \frac{1}{4}$ gives the value 0, i.e. $\tilde{I}_{\text{CHSH}} = 2I_{\text{CHSH}} + 1$, $\tilde{I}_{3322} = I_{3322} + 1$. The results are shown in figure 1.

The software which calculates these maximum quantum mechanical values converges quickly to very consistent results, giving us confidence that they are correct. We see that the new inequality is most important not for states near the maximally entangled state ($\theta = \frac{\pi}{4}$), but rather for states with less symmetry.

It also seems that the CHSH inequality is still relevant: there are states which violate it and which do not violate I_{3322} . This is an illusion, due to the fact that we only maximized the violation over non-degenerate Von-Neumann measurements. Surprisingly, the maximum violation of the new inequality is often given by degenerate measurements. For example, if we take I_{3322} and set $A_3 = 1$ and $B_1 = 1$, the remaining measurements give us the CHSH inequality. Thus given I_{3322} , the CHSH inequality is no longer relevant.

I_{3322} shows a very direct non-locality in the states of equation (29). It is worth noting that such states are also non-local by Popescu’s ‘hidden non-locality’ criterion [17, 18]. In this one first makes local filtrations to the state on both particles, and then performs a standard CHSH test on the state which will emerge if both particles pass the filters. One only looks at the data in the case where both filters are passed, and if this data violates the CHSH inequality we are assured that the original state is non-local. For our states we would apply a local filter to particle A which lets state $|1\rangle$ pass, and absorbs state $|0\rangle$ with high probability. We would simultaneously apply a local filter to particle B which lets $|1\rangle$ pass, and absorbs $|0\rangle$ with high probability. The idea is that each component of the entangled state is only filtered once, whereas the noise term is filtered twice. If both particles pass the filter, the state is very close to a pure entangled state. Since all pure qubit states violate the CHSH inequality [19], this one does too, and we have shown hidden non-locality.

We have also tested the new inequality on the 2-qubit Werner state [21]:

$$\rho_p = pP_{|\psi\rangle} + (1 - p)\frac{I}{4} \tag{30}$$

where $|\psi\rangle$ is the maximally entangled state, as in equation (20). This state is interesting since despite being entangled for $p > \frac{1}{3}$, Werner gave an explicit LHV model for all Von-Neumann measurements for $p \leq \frac{1}{2}$. I_{CHSH} is violated for $p > \frac{1}{\sqrt{2}}$, leaving a region $\frac{1}{2} < p \leq \frac{1}{\sqrt{2}}$ where there may or not be model. If there is not a model, it must be that some Bell inequality n_1n_222 is violated. We find that I_{3322} gives a violation only for $p > \frac{3}{4}$, suggesting that such a model exists.

5. Non-locality sharing

Another important feature of this inequality is that non-locality can be shared between qubits. Imagine that we have three qubits, A, B and C , and we ask whether one can simultaneously give non-local correlations with the second (summing over the third particle’s outcomes), and non-local correlations with the third (summing over the second particle’s outcomes). For CHSH non-locality, the answer is ‘no’ [20]: non-locality is monogamous. We can violate the inequality between parties A and B , or A and C , or have both pairs give the lhv maximum, but never violate both at the same time.

The non-locality shown by the new inequality can be shared. Take the 3-qubit state

$$|\psi\rangle = \mu |000\rangle_{ABC} + \sqrt{\frac{1 - \mu^2}{2}}(|110\rangle_{ABC} + |101\rangle_{ABC}) \tag{31}$$

with $\mu = 0.852$. Qubits B and C are symmetric, and qubits A and B violate I_{3322} giving a value 0.0041. The measurements are defined by the azimuthal and polar angles:

$$A_1 = (\alpha, 2\pi - \beta) \tag{32}$$

$$A_2 = (\alpha, \pi - \beta) \tag{33}$$

$$A_3 = \left(\frac{\pi}{2}, 2\pi - \delta\right) \tag{34}$$

$$B_1 = (\gamma, \pi + \delta) \tag{35}$$

$$B_2 = (\gamma, \delta) \tag{36}$$

$$B_3 = \left(\frac{\pi}{2}, \beta\right) \tag{37}$$

where $\alpha = 2.8252, \beta = 0.1931, \delta = 0.0804$ and $\gamma = 2.5445$.

6. Many measurement inequalities

Inspired by this case, we have found a generalization of this inequality to the $mm22$ case. For 4422 it looks as follows:

$$I_{4422} = \left(\begin{array}{c|cccc} & -1 & 0 & 0 & 0 \\ \hline -3 & 1 & 1 & 1 & 1 \\ -2 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array} \right). \tag{38}$$

The past-determined correlations are always ≤ 0 . The generalization to $mm22$ should now be clear. The main part of the matrix has entries one in every position from the top left corner to the backwards diagonal. There is then one backwards off-diagonal line of -1 entries, and afterwards 0 complete the matrix. We then subtract $\sum_{i=1}^{m_B} (m_B - i)P(B_i) + P(A_1)$.

We shall prove the ltv maximum by induction. Starting from the lower left corner, we can see that I_{mm22} contains all the inequalities from the same family with less measurements. If we just take measurements A_1 and B_m (ignoring the other measurements by setting their outcomes to one), we have a positive probability face. Adding measurements A_2 and B_{m-1} gives the CHSH inequality. Adding A_3 and B_{m-2} gives us I_{3322} . Let us assume that we have proved $I_{(m-1)(m-1)22} \leq 0$. To get a value larger than 0 for I_{mm22} , we must total at least +1 in the terms which were not present in the previous inequality. We can only do this by setting $A_i = 0 \forall i$, and $B_1 = 0$. This gives us +1 in the new terms. But now we have a -1 from $P(A_1)$. Whatever we put for the values of B_j for $j = 2, \dots, m$, each row j contributes exactly 0 to the total, giving us $I_{mm22} = 0$, and proving that this is the maximum.

Unfortunately, we do not know if I_{mm22} is a face for all m . We have found computationally that it is indeed a face for $m \leq 7$, and suspect that this will generalize. We also do not know if any of the other inequalities $mm22$ are relevant after one already has the 3322 inequality. An analytic problem here is that we have no simple criterion to say which states definitely do not violate the 3322 inequality. Even computationally we have not yet found an example of a state which would violate one of the inequalities $mm22$ without violating I_{3322} . Whilst one would expect to find new, inequivalent faces at every m , it is not clear that they will all be relevant, particularly if we fix ourselves to a certain quantum system size, like 2-qubits.

For the case 3422, we find 12480 faces, which include three new inequalities, along with I_{3322} , the CHSH and positive probability faces. As for our I_{mm22} inequalities, we do not know a good way to discover whether these new inequalities are relevant, or to uncover other interesting features they may possess. The new inequalities are in appendix.

We have not gone beyond 3422 in a complete way at present since our software takes too long to run on our PC. We are able to produce a subset of the faces, but have not investigated this direction.

7. Many outcome inequalities

We can also look at inequalities with more measurement outcomes. For 2223 there are no new types of face. There are only the CHSH, and positive probability faces. To use the CHSH inequality (which is defined for two outcome measurements) for three outcome measurements, we map the three outcomes into two effective outcomes by putting two of the original outcomes together. We then put the effective outcomes into the CHSH inequality. This can be done in three different ways for each of particle B measurements. Since there

are eight versions of the CHSH inequality for 2222, this gives $8 * 3^{m_B} = 72$ faces. There are $m_A m_B n_A n_B = 24$ positive probability faces, making 96 faces in total.

For 2224 we again find no new types of faces. There are 32 positive probability faces, and 392 CHSH faces. Note that there are two different ways to put together the four outcomes: we can group three of them together against the 4th, or put them in two groups of 2. There are four ways to do the first, and three to do the second, giving $8 * (4 + 3)^{m_B} = 392$ faces.

For 2225 and 2226 we have computed a list of all the faces, but have not been able to sort them. The number of faces, 1840 and 7736, is that which one predicts assuming there are no new faces. Therefore we conjecture that there are no faces beyond the CHSH for the case 222n.

For the case 2233, there is only one new type of inequality, which is already found in [12, 22]. This can be written as

$$I_{2233} = \left(\begin{array}{c|cc|cc} & -1 & -1 & 0 & 0 \\ \hline -1 & 1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & -1 & -1 \end{array} \right). \tag{39}$$

The columns of the correlation part of the matrix correspond to $A_1 = 0, A_1 = 1, A_2 = 0$ and $A_2 = 1$. The rows are in the same order, for particle B. Thus the first entry is $P(A_1 = 0, B_1 = 0)$. For lhv models $I_{2233} \leq 0$.

The total number of faces for the case 3322 is 1116, of which 36 are positive probability, $8 * 3^{m_A} 3^{m_B} = 648$ are CHSH, and 432 are I_{2233} .

For states of 2-qutrits of the form

$$\rho_p = p P_{|\psi\rangle} + (1 - p) \frac{I}{9} \tag{40}$$

where $|\psi\rangle = \frac{1}{\sqrt{3}}(|0, 0\rangle + |1, 1\rangle + |2, 2\rangle)$, I_{2233} is violated by states with more noise (a smaller p) than the CHSH inequality. Thus it is relevant. On the other hand, we can recover the CHSH inequality from this one by using the outcomes 1 and 2 for measurements A_1 and A_2 , and outcomes 0 and 2 for measurements B_1 and B_2 . Once we have this new inequality, the CHSH is no longer relevant.

This inequality has been generalized to 22nn [12]. The generalized inequalities are known to be faces for all n , and to be the only faces which exist of a certain form [23]. In our present notation, they look simpler than they did in the original paper, for example

$$I_{2244} = \left(\begin{array}{c|ccc|ccc} & -1 & -1 & -1 & 0 & 0 & 0 \\ \hline -1 & 1 & 1 & 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 & -1 & -1 & -1 \end{array} \right). \tag{41}$$

The lhv maximum is 0. To see this, note that to get more, the local terms $-P(B_1 \neq n)$ and $-P(B_2 \neq n)$ force us to try to get +1 from the three pairs of measurements $(A_1, B_1), (A_1, B_2)$ and (A_2, B_1) . But if we do this, we are forced to get a -1 from (A_2, B_2) , leaving us with a total of 0.

The generalization of the inequality to more outcomes is as one would guess.

For 2234 we have computed all the faces, but not sorted them. The total number of faces, 19128, matches the number one expects assuming there are no new inequalities. Beyond this our program would take too long to compute the complete solution.

8. Many measurements and outcomes

What about inequalities combining more measurements and more outcomes? Garg and Mermin [24] have evidence which suggests that such inequalities exist. They found a quantum state and measurements for the case 3333 for which the results satisfy all 2222 and 2233 inequalities, but are nevertheless non-local. We have found a family of inequalities for the case $mmnn$, which is a generalization of the inequalities for $22nn$ and $mm22$. The first member is

$$I_{3333} = \left(\begin{array}{c|ccc} & -\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline -\mathbf{2} & X & X & Y \\ -\mathbf{1} & X & Y & -Y \\ \mathbf{0} & Y & -Y & -Z \end{array} \right) \tag{42}$$

where $X = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $Z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and \mathbf{C} is a row (or column) in which every entry is \mathbf{C} .

$I_{3333} \leq 0$ for past-generated correlations. X and Y are the same matrices which appear in I_{2233} , and the arrangement of X and Y is similar to the arrangement of the elements of the matrix in I_{3322} . Z is a new matrix we put in by hand, because the more natural matrix full of 0's did not give us a face. The local probabilities which we subtract are a natural generalization of the terms from I_{2233} and I_{3322} .

To generate the complete $mmnn$ family we first generalize the number of measurements, then the number of outcomes. Adding one more measurement gives

$$I_{4433} = \left(\begin{array}{c|cccc} & -\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline -\mathbf{3} & X & X & X & Y \\ -\mathbf{2} & X & X & Y & -Y \\ -\mathbf{1} & X & Y & -Y & -Z \\ \mathbf{0} & Y & -Y & -Z & -Z \end{array} \right). \tag{43}$$

The generalization to $mm33$ follows a similar pattern to that for $mm22$. The matrix consists of entries X for all the elements from the top-left corner to just before the main backwards diagonal. The main backwards diagonal has entries Y , and the next backwards diagonal has entries $-Y$. The lower right corner is filled by entries $-Z$. We then subtract $P(A_1 \neq 2) + \sum_{j=1}^{m_B} (m_B - j)P(B_j \neq 2)$.

To generalize to more outcomes, one only has to change the matrices X , Y and Z . X and Y change exactly as they do in the family $22nn$. Z grows in a slightly odd looking manner:

$$Z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{44}$$

which is the same as Y but for the last row.

$I_{mmnn} \leq 0$ for lhv theories. To prove this, we combine the proofs of $I_{mm22} \leq 0$ and $I_{22nn} \leq 0$. Starting at the bottom left corner, and successively adding pairs of measurements, we see that I_{mmnn} contains all the inequalities $I_{m'm'nn}$, with $m' \leq m$. For $m = 1$, the inequality is trivial. For $m = 2$, the inequality is I_{22nn} , which we have already proved. For $m = 3$, to get

more than 0 we need to get a positive contribution from the terms added on after $m = 2$. Thus we must get a +1 from all the combinations (A_1, B_1) , (A_2, B_1) , (A_3, B_1) . But now we have -1 from A_1 . To get a total of more than 0, we need to find a row B_k where the contribution from that row is +1. To see that this is impossible, first look at the row B_2 . We want to pick up +1 from (A_1, B_2) and (A_2, B_2) without picking up a -1 from (A_3, B_2) . But the +1's in (A_2, B_2) , (A_2, B_1) and (A_3, B_1) force the -1 to occur, making the maximum of the row 0. Otherwise look at row B_3 . Here setting $B_3 = n - 2$ is no use, since we get a +1 and a -1 . $B_3 = n - 1$ is clearly useless, also giving us 0. For $B_3 < n - 2$, the matrix $-Z$ looks like $-Y$, and so will always give -1 for (A_3, B_3) when we get +1 for (A_1, B_3) , (A_1, B_1) and (A_3, B_1) . So we have proved $I_{33nn} \leq 0$. For more measurements, a similar argument leads to a proof by induction.

We know computationally that these inequalities are faces for $m = 2, n \leq 7$, for $m = 3, n \leq 6$, for $m = 4, n \leq 4$ and for $m = 5, n = 3$. We suspect this is true for all m and n . As was the case for our family of $mm22$ inequalities, we do not know if these new $mmnn$ inequalities are relevant.

9. Conclusions

In summary we have found that for small numbers of measurements and outcomes there are very few inequivalent Bell inequalities. For the case $2m22$ there is only the CHSH inequality. We believe the same to be true for the case $222n$. For the cases 2233 and 3322 there is the CHSH inequality, but only one other inequality in each case. The new inequalities are relevant: they are violated by states which do not violate the CHSH inequality. The 3322 case shows that the CHSH inequality is not the only useful one for 2-qubits: more measurements really help! We have also discovered a family of inequalities for the case $mmnn$, but do not know if any of these inequalities are relevant.

Having found that there are remarkably few relevant inequivalent faces, we have more time to study closely the ones which do exist. Will some of them help us to perform an experiment definitively ruling out lhv correlations in the laboratory? Is there a close connection between the new inequalities and a particular quantum cryptography protocol? How does this compare with different types of quantum correlations? All these possibilities would be interesting, but here we have a more fundamental message. The set of past-generated correlations is not as complicated as we thought.

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Note Added. C Śliwa has independently found some of the results contained in this paper. Firstly that for the case $2n22$ there are no inequalities beyond the CHSH. Secondly that for the case 3322 there is only a single new inequality (this is, like our result, an exact computational result). He has also investigated the three party case 222222 .

Appendix. New 3422 inequalities

There are 12480 faces for the case 3422 , which include three new faces. There are $3 \cdot 4 \cdot 2 \cdot 2 = 48$ positive probability faces, $8 \binom{3}{2} \binom{4}{2} = 144$ CHSH faces and $576 \binom{4}{3} = 2304 I_{3322}$ faces.

Then the three new inequalities. There are 2304 versions of

$$I_{3422}^1 = \left(\begin{array}{c|ccc} & 1 & 1 & -2 \\ \hline 1 & -1 & -1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ \hline 1 & -1 & -1 & -1 \end{array} \right) \quad (\text{A1})$$

which has a lhv maximum of two. There are 3027 versions of

$$I_{3422}^2 = \left(\begin{array}{c|ccc} & 0 & 1 & -1 \\ \hline -1 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 \\ \hline 1 & -1 & -1 & 0 \end{array} \right) \quad (\text{A2})$$

with a lhv maximum of one. Finally 4608 versions of

$$I_{3422}^3 = \left(\begin{array}{c|ccc} & 1 & 0 & -1 \\ \hline 0 & -2 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ \hline 2 & -1 & -1 & -1 \end{array} \right) \quad (\text{A3})$$

with a lhv maximum of two.

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